To compound money forward in time:

To discount money back in time:

r = annualized interest rate; m = number of compounding periods per year; t = number of years

Negative Interest Rate

Present Value of a Perpetuity

Present Value of a deferred perpetuity

Present value of an annuity

Compounding/Discounting of Bonds

Annually:

Semiannually: = 48.06

Monthly:

General Form:

annualized rate for m compounding frequency

m: compounding frequency

T-t = number of years

Continuous:

General Form Continuous:

Bond Price: +

Annuity Factor: Coupon\*

Ex: = 262.11 + 790.31 = 1052.42

Discrete Returns: 🡺

Continuous Returns: 🡺

Periodic to continuous interest rate formula

or

Discounting a single cashflow assuming continuous compounding

OR

Bond Price or Cashflows assuming continuous compounding

Annual Coupons or Cashflows

Semiannual Coupons or Cashflows

Ex:

\*Continuous compounding formula can be used even though semiannual payments if rate is continuously compounded

Annuity Factor assuming continuous compounding

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Term Structure of Interest Rates** | | | | | | |
| Maturity | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
| Rate: | 0.02 | 0.022 | 0.025 | 0.026 | 0.027 | 0.03 |

Formula for pricing a bond w/Term Structure of Interest Rates

Treasury bond with 3 years left until maturity, a 4% semi-annual coupon and a face value of $1000

+

Price = 19.8 + 19.57 + 19.27 + 18.99 + 18.70 + 932.83 = 1026.17

Discount Factor: The discount factor is the present value of $1 over a particular investment horizon for a particular level of risk.

Zero Coupon Bond Valuation

Multiple Payments Coupon Bonds

Full Price (dirty price or invoice price) = Clean Price + Accrued Interest

Example:

Bootstrapping the Yield Curve

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Issue | coupon | maturity | TTM | Price | Yield | Last Trade | Rate |
| US Treasury | 1.625 | 12/31/2021 | 0.441096 | 100.69 | 0.28% | 7/23/2021 | **.28%** |

=

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Issue | coupon | maturity | TTM | Price | Yield | Last Trade | Rate |
| US Treasury | 1.625 | 12/31/2021 | 0.441096 | 100.69 | 0.28% | 7/23/2021 | .28% |
| US Treasury | 2.125 | 6/30/2022 | .936986 | 101.91 | .23% | 7/23/2021 | .23% |

= .22% ≈.23%

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Issue | coupon | maturity | TTM | Price | Yield | Last Trade | Rate |
| US Treasury | 1.625 | 12/31/2021 | 0.441096 | 100.69 | 0.28% | 7/23/2021 | 0.28% |
| US Treasury | 2.125 | 6/30/2022 | 0.936986 | 101.91 | 0.23% | 7/23/2021 | 0.22% |
| US Treasury | 2.125 | 12/31/2022 | 1.441096 | 102.81 | 0.26% | 7/23/2021 | 0.26% |

+

OR

Zero Coupon Replication

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Issue | Cusip | Coupon | Maturity | TTM | Price | Yield | last trade |
| United States Treasury | 912828xw5 | 1.75 | 6/30/2022 | 0.936986 | 101.54 | 0.20% | 7/23/2021 |

Coupon 1.75/2 = .875

Buy some fraction < 1 of a .44 year bond

(100)

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Issue | coupon | maturity | TTM | Price | Yield | Last Trade | Rate |
| US Treasury | 1.625 | 12/31/2021 | 0.441096 | 100.69 | 0.28% | 7/23/2021 | .28% |
| US Treasury | 2.125 | 6/30/2022 | .936986 | 101.91 | .23% | 7/23/2021 | .23% |

2.125/2 = 1.0625

Buy some fraction > 1 of a .94-year bond such that 100.875 =101.625

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Issue | coupon | maturity | TTM | Price | Yield | Last Trade | Rate |
| US Treasury | 1.625 | 12/31/2021 | 0.441096 | 100.69 | 0.28% | 7/23/2021 | .28% |

Replicating Portfolio

* .99814 of bond B
* δ of bond C

.875 = 1.06 δ(100 + 1.625/2) 🡺

Total Cost of Replication

Buy .9981 June 2022 bonds = .99814 \* 101.91 = 101.72

Sell .0018 Dec 2021 bonds = -.0018 \*100.69 = -.1847

Derivative of the bond price

+T

Macaulay Duration: *Elasticity*

=

or for non-annual compounding:

| |

Modified Duration

m = slope

Modified Duration Use:

**Dollar Duration**: The sensitivity of the bond value –*dollar price—*relative to the change in yield to maturity.

OR, using Macaulay or modified duration

$D ≈ -MD \* B(y)

Ex:

= -6.4944

**Convexity**

PV CF Weight PV CF Weight

Semi-annual compounding

Convexity General Case

Bond Price Approximation

Taylor Series Expansion

x)+x)+x)+…

Bond pricing function B(y) can utilize a 3rd order Taylor series expansion:

y)+ … =

y)+ … =

MD C

Portfolio Convexity

=

19.28

=-.9524 \* $x – 28.57 \* $z = -19.28 \*1984.79

=-.9524 \* $x – 28.57 \* $z = -38,259.9984

= x + z = 1984.79

Z = 1984.79 – x

=-.9524 \* x – 28.57 \* (1984.79 -x) = -38,259.9984

Z = 1984.79-667.99 = 1316.80

**Forward Rate Settlement Formula**

* N = notional amount
* d = fraction of year (day count)
* R = contract rate
* r = reference rate

**Forward Rate Generalized Formula**

=

**Forward Contract Payoffs**

**Long Forward:**

**Short Forward:**

**Forward Contract Value**

**Forward Price**

**General**

**For Bonds**

\*AI = Accrued Interest

**Mortgage Monthly Payment**